Non-Standard Schwinger Fermionic Representation of Unitary Group

Fu-Lin Zhang · Jing-Ling Chen

Abstract The non-standard Schwinger fermionic representation of the unitary group is studied by using n-fermion operators. One finds that the Schwinger fermionic representation of the $U(n)$ group is not unique when $n \geq 3$. In general, based on n-fermion operators, the non-standard Schwinger fermionic representation of the $U(n)$ group can be established in a uniform approach, where all the generators commute with the total number operators. The Schwinger fermionic representation of $U(C_n^m)$ group is also discussed.

1 Introduction

There are many kinds of important mappings between spin systems and multi-boson or multi-fermion systems. They not only essentially simplify the analysis of the problems under consideration, but also help us to understand various aspects of them. The most famous among these mappings are the Schwinger [1, 2], Holstein-Primakoff [3] bosonic representations and the Jordan-Wigner transformations [4]. The first two map an arbitrary spin to a bosonic system, while the third establishes a connection between one-dimensional spin-1/2 lattice and spinless fermions on the same lattice. These representations are very successful in describing magnetism in various quantum systems [5–10]. They also play a significant role in many other contexts. For example, the Schwinger representation has been exploited in quantum optics [11] and in the study of certain classes of partially coherent optical beams [12]. The Jordan-Wigner transformation has also been used to simulate the interacting fermions with quantum computers [13].

F.-L. Zhang (✉) · J.-L. Chen
Theoretical Physics Division, Chern Institute of Mathematics, Nankai University, Tianjin 300071, People's Republic of China
e-mail: flzhang@mail.nankai.edu.cn

J.-L. Chen
e-mail: chenjl@nankai.edu.cn

A lot of work has been done to generalize these mappings in several ways. Chaturvedi and Mukunda have extended the Schwinger representation to the $SU(3)$ case on the primes of that each unitary irreducible representation appears exactly once [14]. Kim has discussed the Schwinger representation based on the mixed sets of creation and annihilation operators of bosonic and fermionic type [15]. The Jordan-Wigner transformation has been generalized to arbitrary spin case and also the fermions were replaced by anyons [16, 17]. In addition, there are some new kinds of constructions of the $SU(2)$ algebra being reported [18, 19].

The purpose of this paper is to study the Schwinger representation of unitary group by using n-fermion operators. The work is organized as follows: In Sect. 2, to make the paper be self-contained, we make a brief review for the standard Schwinger representation (SSR) of unitary group with bosons and fermions. In Sect. 3, we show that the realization of the $U(3)$ group with 3-fermion is not unique, by providing a non-standard Schwinger fermionic representation (NSSFR). In Sect. 4, we develop a uniform approach to establish the NSSFR of the $U(n)$ group based on n-fermion operators ($n \geq 3$), where all the generators commute with the total number operators. In Sect. 5, we discuss the Schwinger fermionic representation of the $U(C_n^m)$ group based on n fermions. Conclusion is made in the last section.

2 Brief Review of SSR

In the standard Schwinger bosonic representation of $SU(2)$ group, the three generators are mapped onto the bilinear form of the bosonic operators as

$$
\begin{aligned}
J_1^b &= \frac{1}{2}(b_1^\dagger b_2 + b_2^\dagger b_1), \\
J_2^b &= \frac{1}{2i}(b_1^\dagger b_2 - b_2^\dagger b_1), \\
J_3^b &= \frac{1}{2}(b_1^\dagger b_1 - b_2^\dagger b_2),
\end{aligned}
\tag{1}
$$

where b_i and $b_i^\dagger$ ($i = 1, 2$) are the annihilation and creation operators of the i-th boson, respectively.

The commutation relations of n independent bosonic operators are

$$
[b_i, b_j^\dagger] = \delta_{ij}, \quad [b_i, b_j] = [b_i^\dagger, b_j^\dagger] = 0.
\tag{2}
$$

Let $Q_{ij}^b = b_i^\dagger b_j$, it is easy to have

$$
[Q_{ij}^b, Q_{kl}^b] = \delta_{jk} Q_{il}^b - \delta_{li} Q_{kj}^b,
\tag{3}
$$

which remarkably shares the same commutation relation of matrices e_{ij}, having 1 in the (i, j) position (i.e., the i-th arrow and the j-th column of the matrix) and 0 elsewhere. If G_i is a matrix generator of a Lie group and $G_i^{\alpha\beta}$ denotes its elements in the (α, β) position, then the operators G_i^b, which are defined by the linear combinations of Q_{ij}^b as

$$
G_i^b = \sum_{\alpha\beta} G_i^{\alpha\beta} Q_{\alpha\beta}^b = \sum_{\alpha\beta} b_\alpha^\dagger G_i^{\alpha\beta} b_\beta,
\tag{4}
$$

obey the same commutation relations as those of the matrices G_i. It means that the operators G_i^b form a representation of the Lie group represented by G_i. When $n = 2$ and $G_i = \sigma_i/2$

(where σ_i are the Pauli matrices, $i = 1, 2, 3$), (4) gives the standard Schwinger bosonic representation of the $U(2)$ group as in (1).

The annihilation and creation operators in (4) are not necessarily restricted to bosons. Notice that the anticommutation relations of n independent fermions are

$$\{a_i, a_j^\dagger\} = \delta_{ij}, \quad \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0, \tag{5}$$

where a_i and $a_i^\dagger$ are annihilation and creation operators of the i-th fermion. The key point is that the bilinear form operators $Q_{ij}^f = a_i^\dagger a_j$ also obey the commutation relations of e_{ij}. Thus replacing $Q_{\alpha\beta}^b$ by $Q_{\alpha\beta}^f$ in (4), one gets the standard Schwinger fermionic representation of the Lie group

$$G_i^f = \sum_{\alpha\beta} G_i^{\alpha\beta} Q_{\alpha\beta}^f = \sum_{\alpha\beta} a_\alpha^\dagger G_i^{\alpha\beta} a_\beta. \tag{6}$$

There are n^2 independent operators Q_{ij}^b or Q_{ij}^f for n bosons or fermions. They can construct the $U(n)$ Lie algebra, and all the generators commute with the total number operator $N^b = \sum_{i=1}^n N_i^b = \sum_{i=1}^n b_i^\dagger b_i$, or $N^f = \sum_{i=1}^n N_i^f = \sum_{i=1}^n a_i^\dagger a_i$. Since the Hamiltonians of the n-dimensional bosonic and fermionic oscillators are $H^b = N^b + \frac{n}{2}$, $H^f = N^f + \frac{n}{2}$. It indicates that both n-dimensional bosonic and fermionic oscillators have the $U(n)$ symmetry.

3 NSSFR of $U(3)$ Group with 3-Fermion

In this section we would like to study the NSSFR of $U(3)$ group with three fermions. For three fermions, there is only one independent 6-order operator $a_1^\dagger a_1 a_2^\dagger a_2 a_3^\dagger a_3$ and no higher-order operator. The 2-order operators have been used to realized the standard Schwinger fermion representation. Here we attend to apply the 4-order operators to establish the NSSFR.

In the fundamental representation [20] of the $U(3)$, the generators are $G_i = \lambda_i/2$, $i = 1, 2, \ldots, 8$, where λ_i are the Gell-Mann matrices. They obey the $U(3)$ algebraic commutation relations

$$[\lambda_i, \lambda_j] = 2i \sum_{k=1}^8 f_{ijk} \lambda_k, \tag{7}$$

where f_{ijk} is the structure constant. Notice that the $j = 1$ representations of the $U(2)$ algebra are

$$J_+^{(1)} = J_1^{(1)} + i J_2^{(1)} = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix},$$

$$J_-^{(1)} = J_1^{(1)} - i J_2^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}, \tag{8}$$

$$J_3^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The Gell-Mann matrices can be constructed by using the quadratic forms of the above spin-1 matrices as

$$\lambda_1 = \frac{\sqrt{2}}{2}(J_3^{(1)} J_+^{(1)} + J_-^{(1)} J_3^{(1)}),$$

$$\lambda_2 = \frac{-i\sqrt{2}}{2}(J_3^{(1)} J_+^{(1)} - J_-^{(1)} J_3^{(1)}),$$

$$\lambda_3 = \frac{1}{2}[J_3^{(1)} J_+^{(1)}, J_-^{(1)} J_3^{(1)}],$$

$$\lambda_4 = \frac{1}{2}(J_+^{(1)2} + J_-^{(1)2}),$$

$$\lambda_5 = \frac{-i}{2}(J_+^{(1)2} - J_-^{(1)2}), \qquad\qquad\qquad (9)$$

$$\lambda_6 = \frac{-\sqrt{2}}{2}(J_3^{(1)} J_-^{(1)} + J_+^{(1)} J_3^{(1)}),$$

$$\lambda_7 = \frac{i\sqrt{2}}{2}(J_3^{(1)} J_-^{(1)} - J_+^{(1)} J_3^{(1)}),$$

$$\lambda_8 = \frac{1}{4\sqrt{3}}[J_+^{(1)2}, J_-^{(1)2}] + \frac{1}{2\sqrt{3}}[J_3^{(1)} J_-^{(1)}, J_+^{(1)} J_3^{(1)}].$$

Consequently, if the spin-1 matrices in (9) are replaced by their standard Schwinger fermionic representations

$$J_+^f = \sqrt{2}(a_1^\dagger a_2 + a_2^\dagger a_3),$$
$$J_-^f = \sqrt{2}(a_2^\dagger a_1 + a_3^\dagger a_2), \qquad\qquad\qquad (10)$$
$$J_3^f = a_1^\dagger a_1 - a_3^\dagger a_3 = N_1 - N_3,$$

a higher-order representation of the $U(3)$ group can then be obtained as

$$\lambda_1^h = (a_1^\dagger a_2 + a_2^\dagger a_1)(1 - N_3) + (a_2^\dagger a_3 + a_3^\dagger a_2)N_1,$$

$$\lambda_2^h = (-ia_1^\dagger a_2 + ia_2^\dagger a_1)(1 - N_3) + (-ia_2^\dagger a_3 + ia_3^\dagger a_2)N_1,$$

$$\lambda_3^h = N_1 - N_2 - 2N_1 N_3 + N_1 N_2 + N_2 N_3,$$

$$\lambda_4^h = (a_1^\dagger a_3 + a_3^\dagger a_1)(1 - 2N_2),$$

$$\lambda_5^h = (-ia_1^\dagger a_3 + ia_3^\dagger a_1)(1 - 2N_2), \qquad\qquad\qquad (11)$$

$$\lambda_6^h = (a_2^\dagger a_3 + a_3^\dagger a_2)(1 - N_1) + (a_1^\dagger a_2 + a_2^\dagger a_1)N_3,$$

$$\lambda_7^h = (-ia_2^\dagger a_3 + ia_3^\dagger a_2)(1 - N_1) + (-ia_1^\dagger a_2 + ia_2^\dagger a_1)N_3,$$

$$\lambda_8^h = (N_1 + N_2 - 2N_3 + 2N_1 N_3 - N_2 N_3 - N_1 N_2)/\sqrt{3}.$$

One may verify that the commutation relations of λ_i^h are the same as in (7) due to (5). All of the operators λ_i^h commute with the total number operator. It is worthy to mention that the non-standard Schwinger representation in (11) does not valid for bosons.

4 NSSFR with *n*-Fermion

The above method of getting the NSSFR of $U(3)$ group is not conveniently applicable for the arbitrary $U(n)$ group. Thus we would like to rewrite (11) into a uniform form, so that it can be directly generalized to an arbitrary $U(n)$ group.

Let us look at the corresponding matrix-representation of (11) in the occupation number space, whose standard basis reads

$$\{1,\ a_1^\dagger,\ a_2^\dagger,\ a_3^\dagger,\ a_1^\dagger a_2^\dagger,\ a_1^\dagger a_3^\dagger,\ a_2^\dagger a_3^\dagger,\ a_1^\dagger a_2^\dagger a_3^\dagger\}\,|vac\rangle, \tag{12}$$

where $|vac\rangle$ is the vacuum state. Based on (12), the operators in (11) correspond the following 8×8 partitioned matrices:

$$\lambda_i^h = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_i & 0 & 0 \\ 0 & 0 & \lambda_i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{13}$$

where λ_i denotes the i-th Gell-Mann matrix of $U(3)$. Evidently, matrices λ_i^h satisfy the commutation relations of $U(3)$ algebra because matrices λ_i do so. Similarly, the standard Schwinger fermionic representation $\lambda_i^f = \sum_{\alpha=1,\beta=1}^{3,3} a_\alpha^\dagger \lambda_i^{\alpha\beta} a_\beta$ correspond to the following matrices

$$\lambda_i^f = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_i & 0 & 0 \\ 0 & 0 & \lambda_i' & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{14}$$

where $\lambda_i' = U(-\lambda_i^*)U^\dagger$, λ_i^* is the conjugate matrix of λ_i, and U denotes the unitary matrix

$$U = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{15}$$

In the $U(3)$ case, the occupation number space is divided into four invariant subspaces specified by the total particle number, which runs from 0 to 3. Generally, the occupation number space of $U(n)$ group is divided into $n + 1$ invariant subspaces, where the total particle number N runs from 0 to n, and the subspace with $N = m$ is conjugated to the subspace with $N = n - m$.

One may define a selective function

$$f_n^{(m)}(x) = \prod_{i=1}^{m-1} \frac{x-i}{m-i} \prod_{i=m+1}^{n-1} \frac{x-i}{m-i}. \tag{16}$$

$f_n^{(m)}(m) = 1$, and $f_n^{(m)}(x) = 0$ when x equals to any other integer between 1 and $n-1$. Then the NSSFR of $U(3)$ in (11) can be recast to a very simple form

$$\lambda_i^h = \sum_{\alpha,\beta=1}^{n=3} [a_\alpha^\dagger \lambda_i^{\alpha\beta} a_\beta f_3^{(1)}(N) + a_\alpha^\dagger \lambda_i'^{\alpha\beta} a_\beta f_3^{(2)}(N)], \tag{17}$$

where $N = \sum_{i=1}^{n} a_i^\dagger a_i$ denotes the total particle number, and $f_3^{(1)}(N) = -N + 2$ and $f_3^{(2)}(N) = N - 1$ respectively.

Based on (17), the NSSFR of $U(n)$ group can be directly obtained as

$$\lambda_i^h = \sum_{\alpha,\beta=1}^{n} [a_\alpha^\dagger \lambda_i^{\alpha\beta} a_\beta f_n^{(1)}(N) + a_\alpha^\dagger \lambda_i^{\prime\alpha\beta} a_\beta f_n^{(n-1)}(N)], \tag{18}$$

where $\lambda_i' = U(-\lambda_i^*)U^\dagger$, the matrix elements of U are $U_{m,n+1-m} = (-1)^{m+1}$ and the others are zeros.

5 Representation of $U(C_n^m)$ Group with n-Fermion

In the n-fermion occupation number space, the dimension of the subspace with $N = m$ is $C_n^m = n!/m!(n-m)!$. In such a subspace we can construct the Schwinger fermionic representation of an $U(C_n^m)$ group. The largest one we can construct is $U(C_n^{[\frac{n}{2}]})$ group, where $[\frac{n}{2}]$ denotes the integer part of $n/2$.

Let us take the 4-fermion case as an example. The subspace with the total number $N = 2$ is $\{a_1^\dagger a_2^\dagger, a_1^\dagger a_3^\dagger, a_1^\dagger a_4^\dagger, a_2^\dagger a_3^\dagger, a_2^\dagger a_4^\dagger, a_3^\dagger a_4^\dagger\}|vac\rangle$, where $a_i^\dagger$ is the i-th fermionic creation operator. We introduce the notations as $O_1 = a_2 a_1$, $O_2 = a_3 a_1$, $O_3 = a_4 a_1$, $O_4 = a_3 a_2$, $O_5 = a_4 a_2$, $O_6 = a_4 a_3$ and $|i\rangle = O_i^\dagger|vac\rangle$. Then the operators Q_{ij}, which behave as the matrices e_{ij}, in the subspace are

$$Q_{ij} = |i\rangle\langle j| = O_i^\dagger|vac\rangle\langle vac|O_j = O_i^\dagger O_j f_4^{(2)}(N), \tag{19}$$

where the selective function $f_4^{(2)}(N) = -N^2 + 4N - 3$.

The commutation relations among Q_{ij} are the same as those of e_{ij}, i.e., $[Q_{ij}, Q_{kl}] = \delta_{jk} Q_{il} - \delta_{li} Q_{kj}$. All of the 36 Q_{ij}'s commute with the total number operator. If G_i denotes the fundamental representation of i-th generator of the $U(6)$ Lie group, then its Schwinger fermionic representation can be realized as

$$G_i^h = \sum_{\alpha=1,\beta=1}^{6,6} G_i^{\alpha\beta} Q_{\alpha\beta} = \sum_{\alpha=1,\beta=1}^{6,6} O_\alpha^\dagger G_i^{\alpha\beta} O_\beta f_4^{(2)}(N). \tag{20}$$

In general, one can construct the $U(C_n^m)$ Lie algebra using n-fermion operators. To derive the mapping, we first define a set n-dimensional vectors $\zeta = (\zeta_1, \zeta_2 \ldots, \zeta_n)$ where each $\zeta_i = 0$ or 1. There are 2^n such vectors, and C_n^m of them satisfy $\sum_{i=1}^{n} \zeta_i = m$ for a fixed integer m, where $0 \le m \le n$. Label the C_n^m vectors in the descending order, e.g. the first one is

$$\zeta^{(1)} = (\zeta_1^{(1)} = 1, \ldots, \zeta_m^{(1)} = 1, \zeta_{m+1}^{(1)} = 0, \ldots, \zeta_n^{(1)} = 0). \tag{21}$$

Due to $\zeta^{(i)}$, we can define a set of operators as

$$O_i = a_n^{\zeta_n^{(i)}} a_{n-1}^{\zeta_{n-1}^{(i)}} \ldots a_1^{\zeta_1^{(i)}}. \tag{22}$$

The element operators are similarly defined as $Q_{ij} = O_i^\dagger O_j f_n^{(m)}(N)$. Their commutation relations satisfy $[Q_{ij}, Q_{kl}] = \delta_{jk} Q_{il} - \delta_{li} Q_{kj}$. Then the Schwinger fermionic representations

of the $U(C_n^m)$ group can be realized by

$$G_i^m = \sum_{\alpha,\beta} G_i^{\alpha\beta} Q_{\alpha\beta}^m, \tag{23}$$

where G_i is the i-th $C_n^m \times C_n^m$ matrix generator of the $U(C_n^m)$ group, and $G_i^{\alpha\beta}$ is its element in (α, β) position.

Let $\bar{m} = n - m$, $C_n^{\bar{m}} = C_n^m$. The subspaces with $N = m$ and $N = \bar{m}$ are mutually conjugate. The linear combinations of the operators $Q_{ij}^{\bar{m}}$

$$G_i^{\bar{m}} = \sum_{\alpha,\beta} G_i^{\alpha\beta} Q_{\alpha\beta}^{\bar{m}} \tag{24}$$

also obey the commutation relations of $U(C_n^m)$ algebra. Therefore, when $\bar{m} \neq m$, the more general representation of $U(C_n^m)$ are given by

$$G_i^h = \sum_{\alpha\beta} G_i^{\alpha\beta} Q_{\alpha\beta}^{(m)} \xi_- + \sum_{\alpha\beta} G_i'^{\alpha\beta} Q_{\alpha\beta}^{(\bar{m})} \xi_+, \tag{25}$$

where $\xi_\pm = 0$ or 1, $\xi_+ + \xi_- \neq 0$, G and G' denotes two $C_n^m \times C_n^m$ matrix representations of the $U(C_n^m)$ group.

6 Conclusion

In conclusion, we have studied the non-standard Schwinger fermionic representation of the unitary group by using n-fermion operators. we found that the Schwinger fermionic representation of the $U(n)$ group is not unique when $n \geq 3$. In general, based on n-fermion operators, the NSSFR of the $U(n)$ group can be established in a uniform approach, where all the generators commute with the total number operators. The Schwinger fermionic representation of $U(C_n^m)$ group is also discussed.

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